Chapter 9:
Differential Equations
Really Sections 1.3 and 1.4

Section 9.2: Direction Fields and Euler's Method

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Direction Fields and

Euler's Method

Sec. 9.2: **Direction Fields** & Euler's Method

Idea:

- What if you have a DE that can't be solved by hand?

Sec. 9.2: **Direction Fields** & Euler's Method

Idea:

- What if you have a DE that can't be solved by hand?
- If the DE is of the form $\frac{dy}{dx} = F(x, y)$, we can draw a graph that can help to visualize the solutions. This is called a **Direction Field**.

Sec. 9.2: **Direction Fields** & Euler's Method

Idea:

- To draw a direction field…

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Idea:

- To draw a direction field…
  - For each point in the plane, calculate the slope of the tangent line if the curve were to pass through that point
  - Draw a small line segment through that point with that slope
  - Do this for as many points as possible. (In theory, this is done for EVERY point in the plane)

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Note on Slopes

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Ex 1: Draw a direction field for the DE $\frac{dy}{dx} = x + y$

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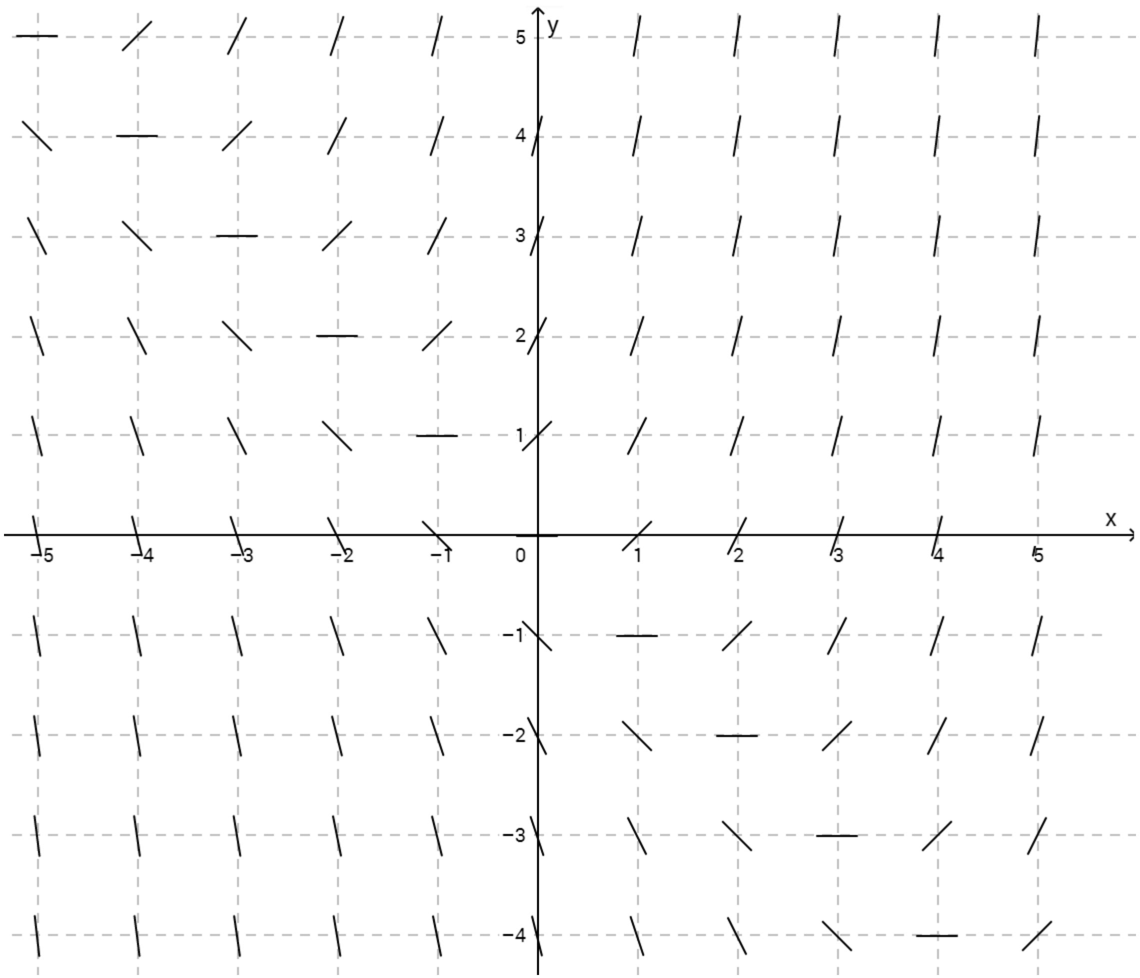
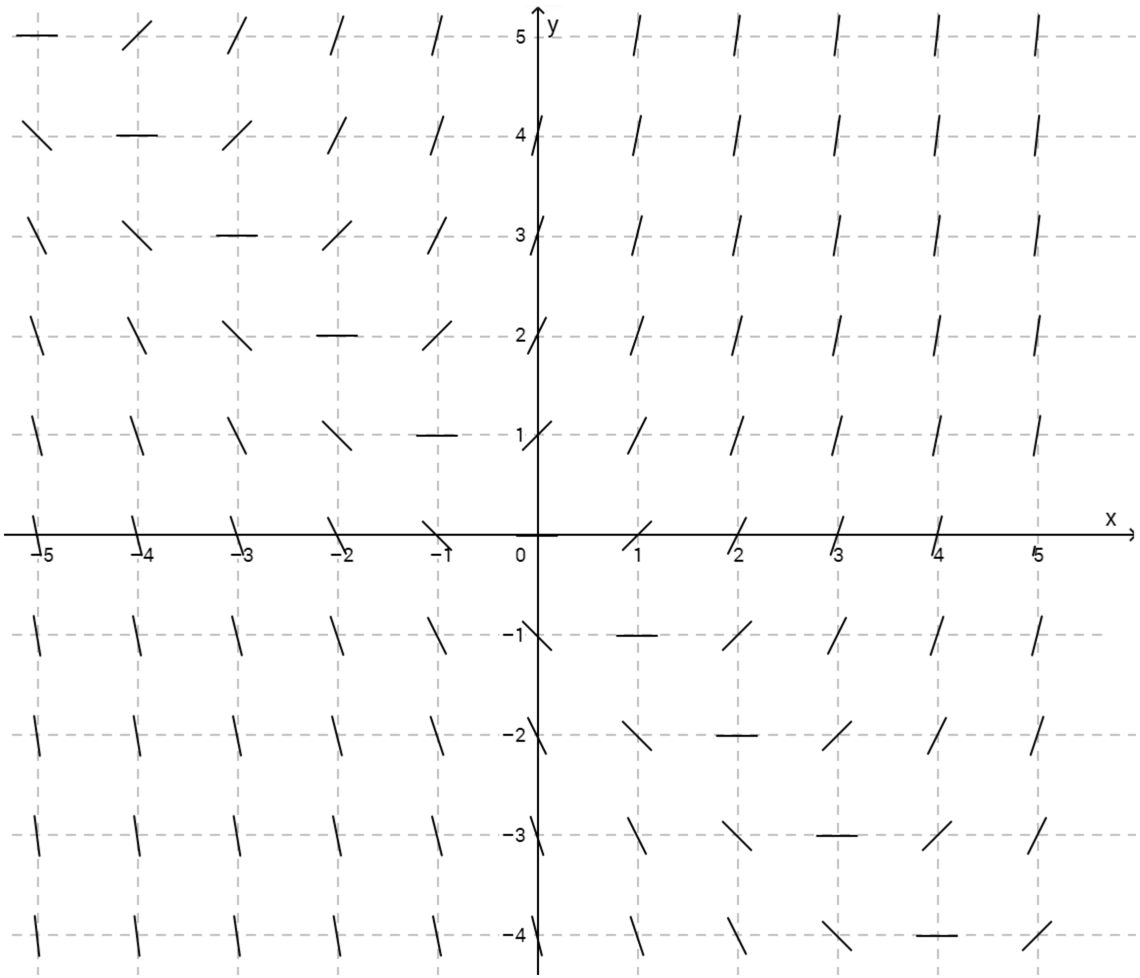


Ex 2: Sketch the graph of the solution to the IVP

$$\frac{dy}{dx} = x + y\ ,\ \ y(0) = 1.$$



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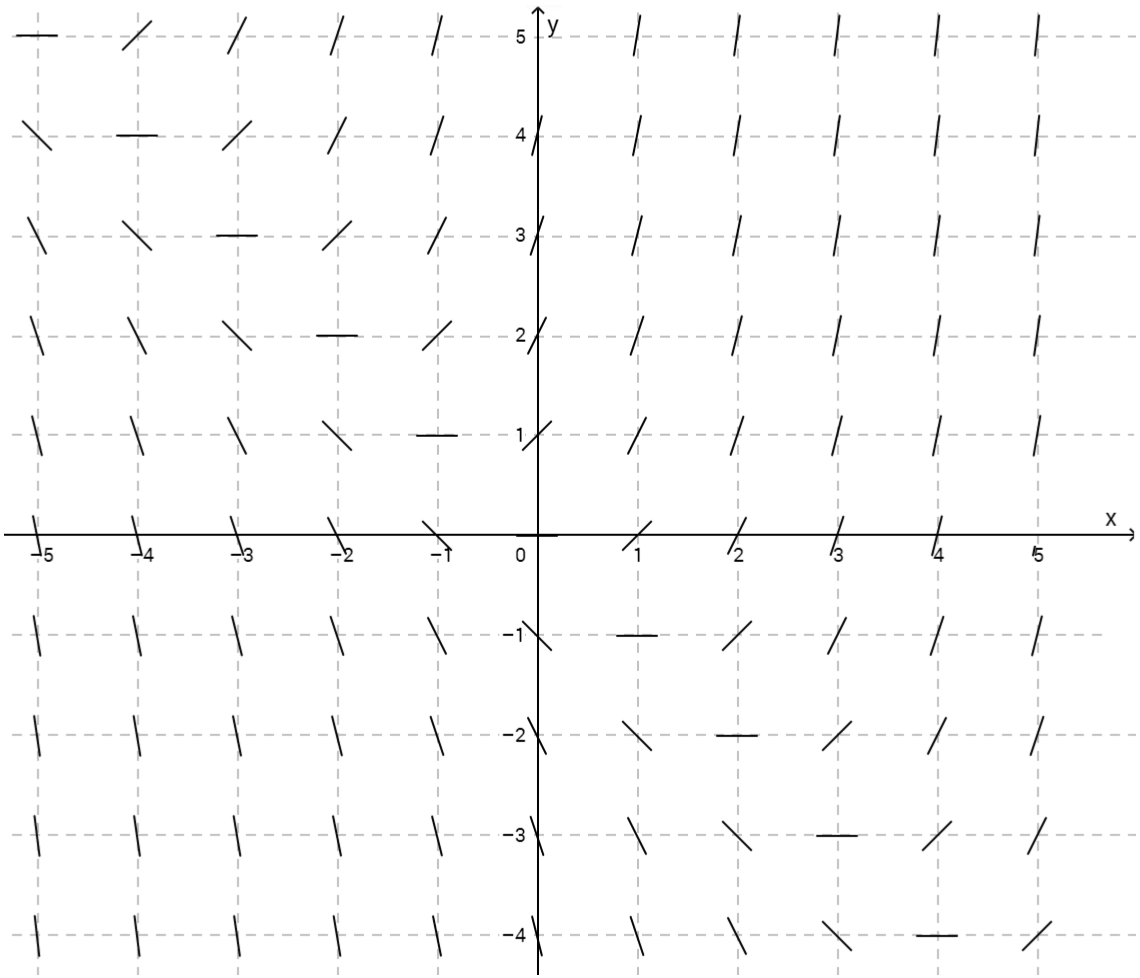
Ex 2: Sketch the graph of the solution to the IVP

$$\frac{dy}{dx} = x + y\ ,\ \ y(-1) = 0.$$



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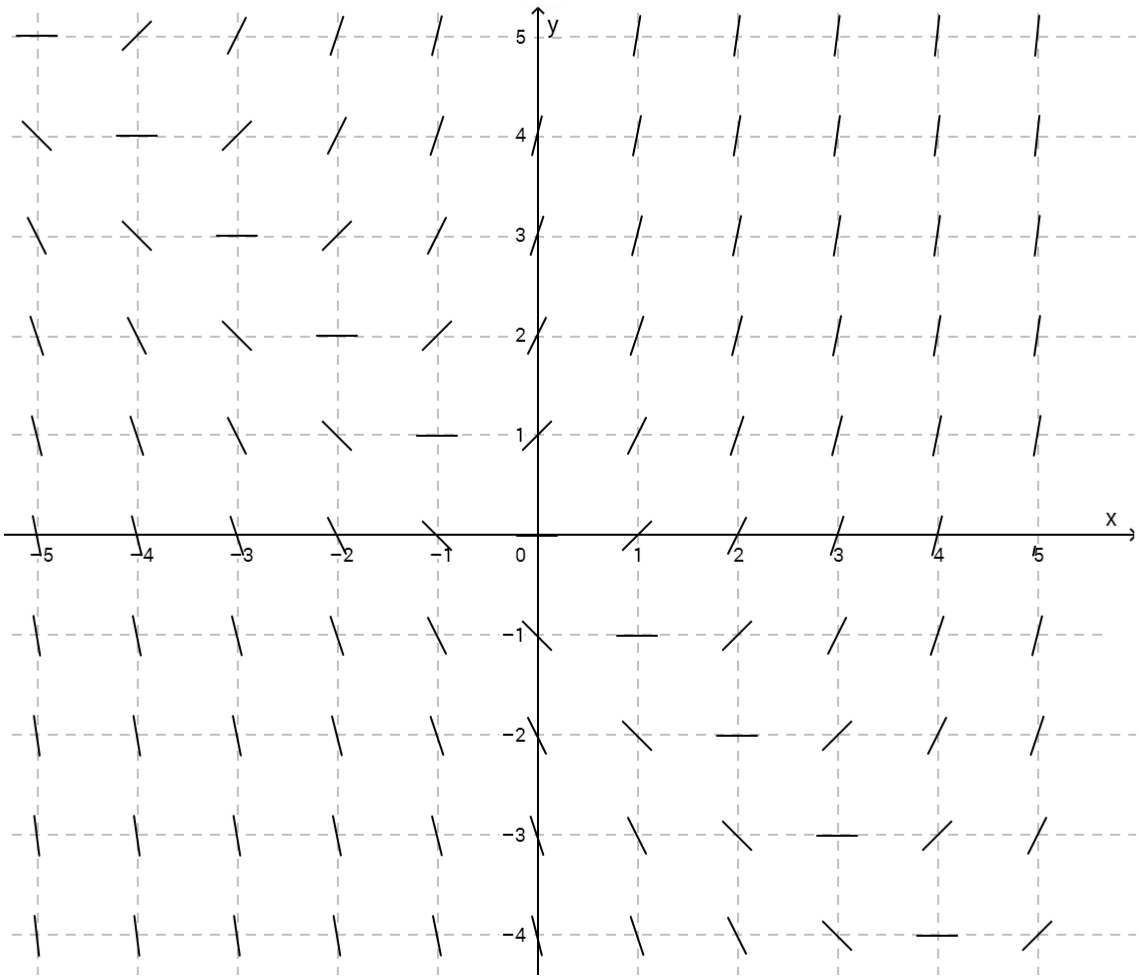
Ex 2: Sketch the graph of the solution to the IVP

$$\frac{dy}{dx} = x + y\ ,\ \ y(-4) = 2.$$



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Def : An equilibrium solution to a DE is a solution of the form $y = \#$. I.e. a solution whose graph is a horizontal line.

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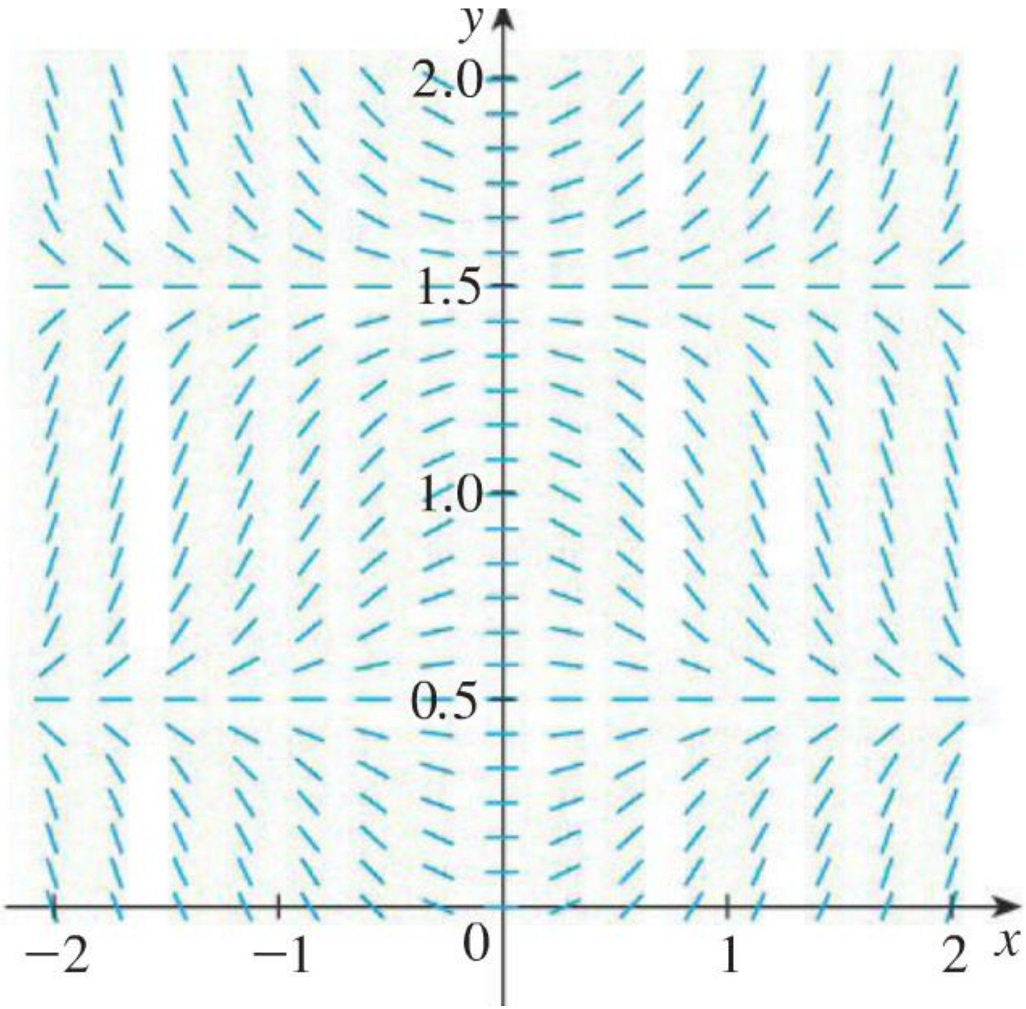
Ex 3: A direction field for the DE $y' = x\cos(\pi y)$ is shown below. Find the equilibrium solutions.



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Ex 4: Find the equilibrium solutions for the following DEs…

a) $\frac{dy}{dx} = x^2 + y^2 - 1$

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b) $\frac{dy}{dx} = xy^3 - 4xy$

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Ex 4: Find the equilibrium solutions for the following DEs…

c) $\frac{dy}{dx} = y - x$

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Sec. 9.2: Direction Fields & **Euler's Method**

Idea: Euler's method is a way of coming up with an approximate solution to an IVP. The solution you get can be either…

1. A list of points that are "close" to points on the actual solution curve, or
2. A curve made up of line segments that is an approximate "solution" to the IVP
3. Once again, the DE must be of the form $\frac{dy}{dx} = F(x, y)$ in order to apply Euler's method

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Another note on slopes:

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Idea behind Euler's Method:

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How Euler's method works:

1. You are given an IVP, so you are given a starting point on the curve. Call this point (x_0, y_0).
2. To get the next point "on the curve", call this point (x_1, y_1) , you must decide on a step-size. The step-size is the distance between the x-coordinates of all the points you are going to find. Call the step-size h.

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How Euler's method works:

3. $x_1 = x_0 + h$

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3. $x_1 = x_0 + h$
4. You get the y-coordinate of this second point by starting at the 1st point (x_0, y_0) then moving in the direction of the direction field until you get to the x-coordinate of the 2nd point. The y-coordinate of this new point is y_1. $y_1 = y_0 + hF(x_0, y_0)$.

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How Euler's method works:

5. In general, suppose besides the point (x_0, y_0) given in the IVP you want n more points on the approximate solution curve.

   The x-coordinates of these points are…
   $$x_1 = x_0 + h\,, \quad x_2 = x_1 + h\,, \quad \ldots \quad x_n = x_{n-1} + h$$

   And the y-coordinates of these points are…
   $$y_1 = y_0 + hF(x_0, y_0)\,, \quad y_2 = y_1 + hF(x_1, y_1)\,, \ldots$$
   $$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

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How Euler's method works:

6. If you plot these points and connect them with straight line segments, you will get an approximation to the solution curve to the IVP made up of line segments.

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How Euler's method works:

6. If you plot these points and connect them with straight line segments, you will get an approximation to the solution curve to the IVP made up of line segments.
7. You will get a better approximation to the actual solution curve by making the step-size smaller.

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Ex 5: Use Euler's Method with a step-size of 0.5 to compute the approximate y values y_1, y_2, y_3, and y_4 of the solution of the initial value problem $y' = y - 2x$, $y(0) = 1$.

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Ex 6: Use Euler's Method with a step-size of 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution to the initial value problem $y' = y + xy$, $y(0) = 1$.

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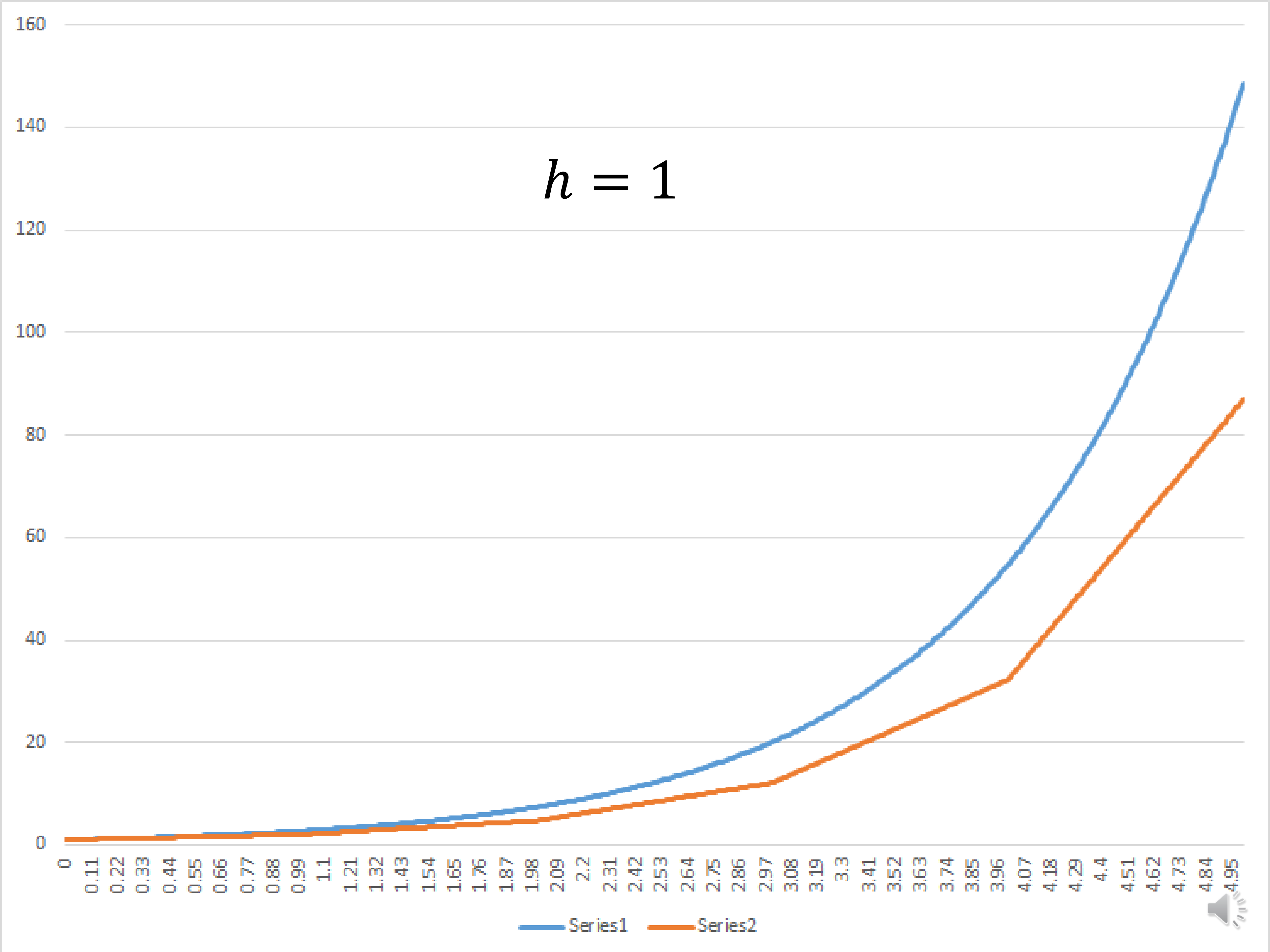
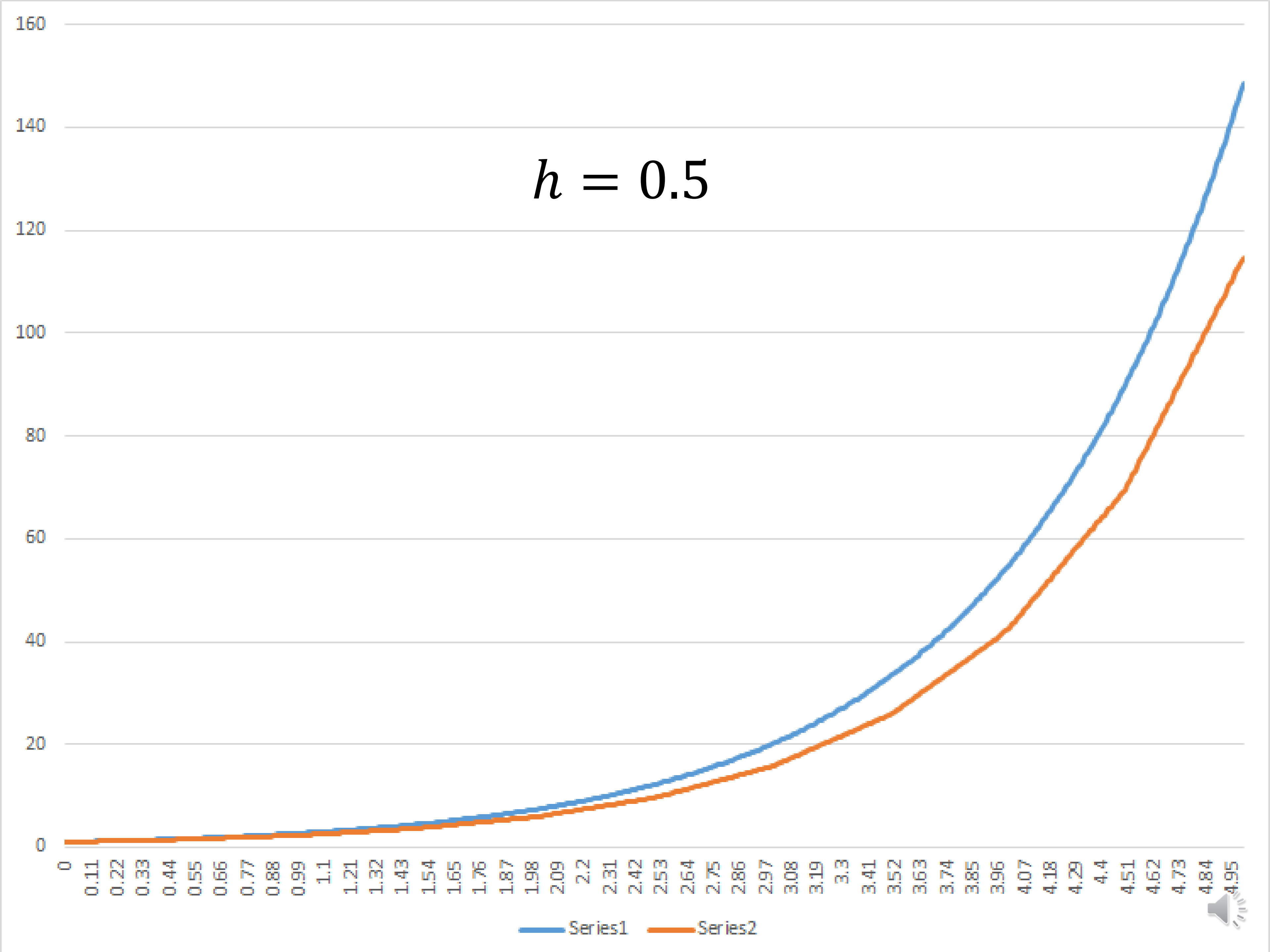
For the IVP $y' = y$, $y(0) = 1$, we know that the solution is $y = e^x$. Use Euler's Method with various step-sizes to see how well the approximation is to the exact solution curve. (We'll look at solutions on [0, 5])



$h = 1$
160
140
120
100
80
60
40
20
0
0 0.11 0.22 0.33 0.44 0.55 0.66 0.77 0.88 0.99 1.1 1.21 1.32 1.43 1.54 1.65 1.76 1.87 1.98 2.09 2.2 2.31 2.42 2.53 2.64 2.75 2.86 2.97 3.08 3.19 3.3 3.41 3.52 3.63 3.74 3.85 3.96 4.07 4.18 4.29 4.4 4.51 4.62 4.73 4.84 4.95
Series1
Series2



$h = 0.5$
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0.33
0.44
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0.77
0.88
0.99
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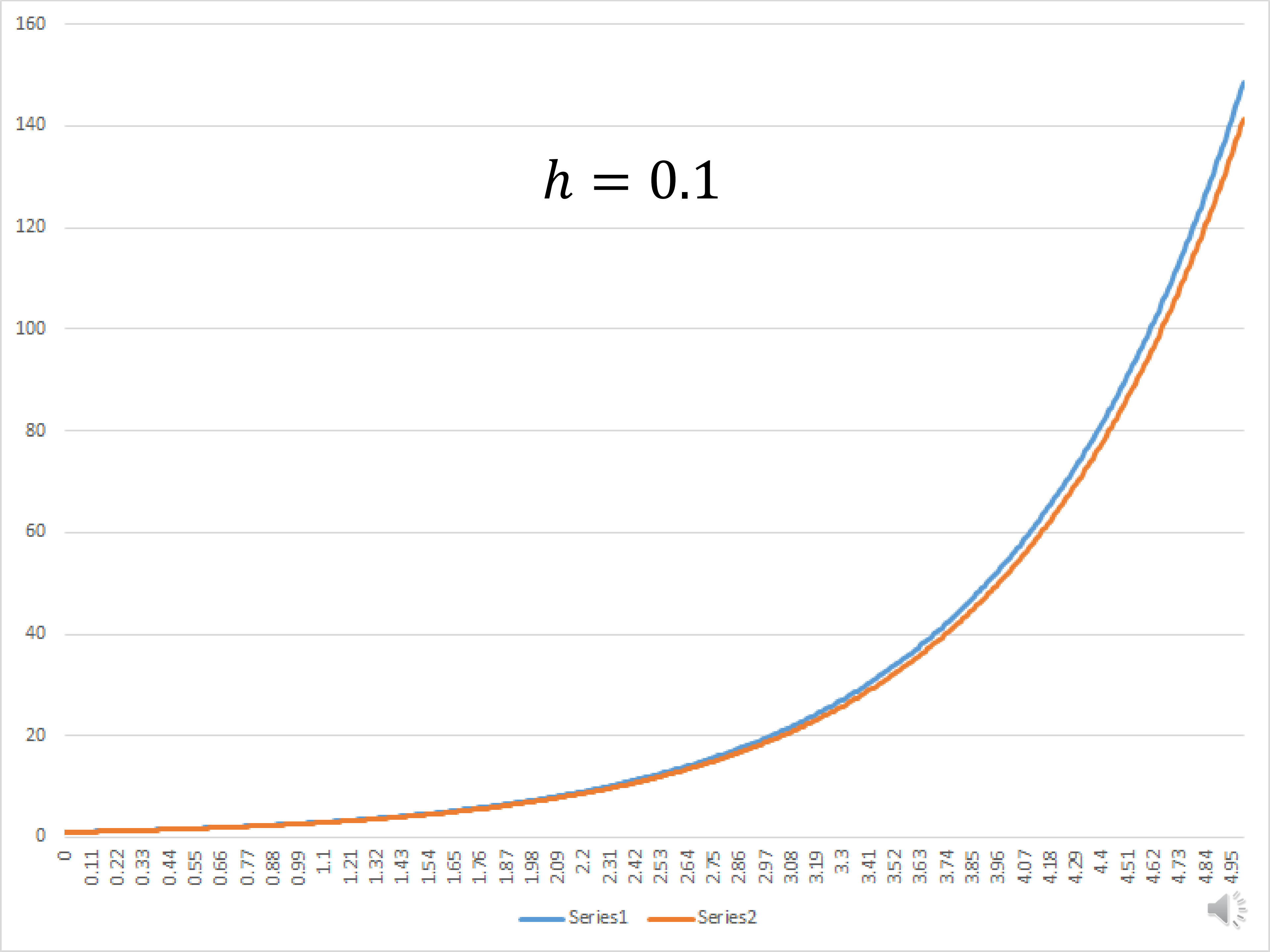

$h = 0.1$
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3.19
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